ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 106 - Business Mathematics II

2018-2019 Spring

SECOND MIDTERM EXAMINATION

25.04.2019, 17:30

- SOLUTIONS -

Question	Grade	Out of
1		24
2		12
3		10
4		12
5		22
6		26
Total		106

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 90 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Evaluate the following integrals ;

(12 pts.) **a)** $\displaystyle\int_0^1 x^5\sqrt{1-x^2}\,dx = \int_0^1 \underbrace{(x^2)^2}_{(1-u)^2} \cdot \underbrace{\sqrt{1-x^2}}_{\sqrt{u}}\ \underbrace{x\,dx}_{-\frac{1}{2}du}$

$u = 1-x^2 \Rightarrow x^2 = 1-u$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$x=0 \Rightarrow u = 1-0^2 = 1$

$x=1 \Rightarrow u = 1-1^2 = 0$

$$= \int_{u=1}^{u=0} (1-2u+u^2)(u^{1/2})\left(-\frac{1}{2}du\right)$$

$$= +\frac{1}{2}\int_0^1 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{3/2}}{\frac{3}{2}} - 2\frac{u^{5/2}}{\frac{5}{2}} + \frac{u^{7/2}}{\frac{7}{2}}\Bigg|_0^1\right] = \frac{1}{2}\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) = \frac{1}{2}\left(\frac{16}{105}\right) = \frac{16}{210} = \boxed{\frac{8}{105}}$$

(12 pts.) **b)** $\displaystyle\int \frac{(x^2+4x)\sqrt{\ln(x^3+6x^2+1)}}{(x^3+6x^2+1)}dx$ (Hint: Let $t = x^3+6x^2+1$)

$t = x^3+6x^2+1$

$dt = (3x^2+12x)\,dx$

$dt = 3(x^2+4x)\,dx$

$\frac{1}{3}dt = (x^2+4x)\,dx$

$$= \int \frac{\sqrt{\ln t}}{t}\left(\frac{1}{3}dt\right) = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\frac{u^{3/2}}{\frac{3}{2}} + C$$

$\left(\begin{array}{l} u = \ln t \\ du = \frac{1}{t}dt \end{array}\right)$

$$= \frac{2}{9}(\ln t)^{3/2} + C = \boxed{\frac{2}{9}\left(\sqrt{\ln(x^3+6x^2+1)}\right)^3 + C}$$

(12 pts.) **2.** Find y subject to the given conditions:

$$y''' = 2x, \quad y''(-1) = 3, \quad y'(3) = 10, \quad y(0) = 13$$

$$\underbrace{\int y'''(x)\,dx}_{y''(x)} = \int 2x\,dx = x^2 + C_1 \Rightarrow \underbrace{y''(-1)}_{3} = (-1)^2 + C_1 = 1 + C_1 \Rightarrow C_1 = 2$$

$$\Rightarrow y''(x) = x^2 + 2 \Rightarrow \underbrace{\int y''(x)\,dx}_{y'(x)} = \int (x^2+2)\,dx = \frac{x^3}{3} + 2x + C_2$$

$$\underbrace{y'(3)}_{10} = \frac{3^3}{3} + 2(3) + C_2 = 9 + 6 + C_2$$

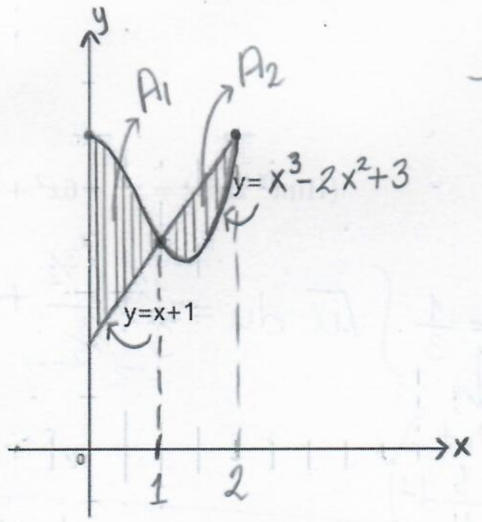
$$C_2 = 10 - 15 = -5$$

$$\Rightarrow y'(x) = \frac{x^3}{3} + 2x - 5 \Rightarrow \underbrace{\int y'(x)\,dx}_{y(x)} = \int \left(\frac{x^3}{3} + 2x - 5\right)dx = \frac{x^4}{12} + x^2 - 5x + C_3$$

$$\underbrace{y(0)}_{13} = C_3 \Rightarrow C_3 = 13$$

$$\Rightarrow \boxed{y(x) = \frac{x^4}{12} + x^2 - 5x + 13}$$

(10 pts.) **3.** Express the area of the below shaded region in terms of an integral(or sum of integrals). DO NOT EVALUATE the integral(s)!



Intersection points:

$$x^3 - 2x^2 + 3 = x + 1 \Rightarrow x^3 - 2x^2 - x + 2 = 0$$

$\boxed{x = 1 \text{ is a soln.}}$

$$\begin{array}{l|l} x^3 - 2x^2 - x + 2 & x - 1 \\ \hline -\; x^3 \pm x^2 & x^2 - x - 2 \\ \quad -x^2 - x & \\ \quad \mp x^2 \pm x & \\ \qquad -2x + 2 & \\ \qquad \pm 2x \mp 2 & \\ \qquad\quad 0 \quad 0 & \end{array}$$

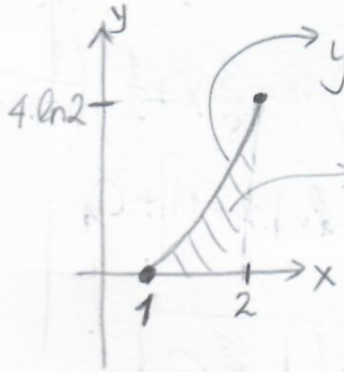
$$\underbrace{x^2 - x - 2}_{0} \Rightarrow (x-2)(x+1) = 0$$

$\boxed{x = 2 \;\&\; x = -1}$

So at $x=1$ & $x=2$ we have intersection pts.

$$A = A_1 + A_2 = \int_0^1 \left[(x^3 - 2x^2 + 3) - (x+1)\right]dx + \int_1^2 \left[(x+1) - (x^3 - 2x^2 + 3)\right]dx$$

(12 pts.) **4.** Find the area of the region bounded by the x-axis and the curve $y = x^2 \ln x$ between $x = 1$ and $x = 2$. (Note: You do not need to sketch the region!)



$y = x^2 . \ln x$

$$A = \int_1^2 \underbrace{x^2 \ln x \, dx} = \left(\frac{x^3}{3} . \ln x\right)\Big|_1^2 - \frac{1}{3}\int_1^2 x^2 \, dx$$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x^2 dx \Rightarrow v = \frac{x^3}{3}$

$$= \left(\frac{8}{3}\ln 2 - \frac{1}{3}\underbrace{\ln 1}_{0}\right) - \frac{1}{3}\,\frac{x^3}{3}\Big|_1^2$$

$$= \frac{8}{3}\ln 2 - \frac{1}{9}(8-1) = \boxed{\frac{8}{3}\ln 2 - \frac{7}{9}}$$

5. Find the integrals ;

(10 pts.) **a)** $\int_1^2 4xe^{2x} dx$ $= 2\int_1^2 \underset{u}{x} . \underbrace{e^{2x} . 2dx}_{dv} = 2\left[\left(xe^{2x}\right)\Big|_1^2 - \int_1^2 e^{2x} dx\right]$

$u = x \Rightarrow du = dx$

$dv = e^{2x} . 2dx \Rightarrow v = e^{2x}$

$$= 2\left[\left(2e^4 - 1e^2\right) - \frac{e^{2x}}{2}\Big|_1^2\right]$$

$$= 2\left[2e^4 - e^2 - \frac{e^4}{2} + \frac{e^2}{2}\right] = \boxed{3e^4 - e^2}$$

(12 pts.) **b)** $\int \frac{3x^3}{\sqrt{4-x^2}} dx$ $= 3\int \underset{u}{x^2} . \underbrace{\frac{x}{\sqrt{4-x^2}} dx}_{dv} =$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \frac{x}{\sqrt{4-x^2}} dx \Rightarrow$

$v = -\frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}} dx$

$t = 4 - x^2$ $= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2} dt$

$= -\frac{1}{2}\,\frac{\sqrt{t}}{\frac{1}{2}} = -\sqrt{4-x^2}$

$$= 3\left[-x^2 . \sqrt{4-x^2} - \int \underbrace{\sqrt{4-x^2}}_{\sqrt{t}}\,\underbrace{(-2x)dx}_{dt}\right]$$

$$= -3x^2 . \sqrt{4-x^2} - 3\,\frac{\left(\sqrt{4-x^2}\right)^3}{\frac{3}{2}} + C$$

$$= \boxed{-3x^2 . \sqrt{4-x^2} - 2\left(\sqrt{4-x^2}\right)^3 + C}$$

6. Determine the integrals ;

(13 pts.) **a)** $\displaystyle\int \frac{3x^3-3x+4}{4x^2-4}dx = \int\left(\frac{3}{4}x + \frac{\cancel{4}}{\cancel{4}(x^2-1)}\right)dx = \frac{3}{8}x^2 + \int\frac{1}{(x-1)(x+1)}dx$

$$3x^3-3x+4 \;\Big|\; 4x^2-4$$
$$\underline{-3x^3 \pm 3x} \qquad \frac{3}{4}x$$
$$0+0+4$$

$$= \frac{3}{8}x^2 + \int\left(\frac{A}{x-1}+\frac{B}{x+1}\right)dx = \frac{3}{8}x^2 + \frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$$

$$\frac{1}{(x-1)(x+1)} = \frac{(A+B)x+(A-B)}{(x-1)(x+1)}$$

$$= \frac{3}{8}x^2 + \frac{1}{2}\Big[\ln|x-1| - \ln|x+1|\Big] + C_1$$

$$\left.\begin{array}{l} A+B=0 \\ A-B=1 \end{array}\right\} \quad 2A=1 \Rightarrow A=\frac{1}{2}, \quad B=-\frac{1}{2}$$

$$= \boxed{\frac{3}{8}x^2 + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C_1}$$

(13 pts.) **b)** $\displaystyle\int \frac{7x^3+24x}{(x^2+3)(x^2+4)}dx = \int\left(\frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2+4}\right)dx = \int\left(\frac{3x}{x^2+3}+\frac{4x}{x^2+4}\right)dx$

$$7x^3+24x = x^3(A+C)+x^2(B+D)+x(4A+3C)+(4B+3D)$$

x^3: $7 = A+C$

x^2: $0 = B+D$

x: $24 = 4A+3C$

con.: $0 = 4B+3D$

$\rightarrow B=D=0$

$$\left.\begin{array}{l} A+C=7 \\ 4A+3C=24 \end{array}\right\} \quad A=3,\; C=4$$

$$= \frac{3}{2}\ln|x^2+3| + \frac{4}{2}\ln|x^2+4| + C_2$$

$$= \boxed{\ln\left[(x^2+3)^{3/2}\cdot(x^2+4)\right] + C_2}$$