ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 106 - Business Mathematics II

2018-2019 Spring

FIRST MIDTERM EXAMINATION

19.03.2019, 17:30

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 90 minutes

Question	Grade	Out of
1		8
2		14
3		24
4		19
5		20
6		21
Total		106

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. (8 pts) Construct a 4×4 matrix $A = [a_{ij}]$ whose entries satisfy $\mathbf{a_{ij}} = \mathbf{i^{j-1}}$, $i, j = 1, 2, 3, 4$.

$$\begin{cases} a_{11} = 1^{1-1} = 1^0 = 1, \ a_{12} = 1^{2-1} = 1^1 = 1, \ a_{13} = 1^{3-1} = 1^2 = 1, \ a_{14} = 1^{4-1} = 1^3 = 1 \\ a_{21} = 2^{1-1} = 2^0 = 1, \ a_{22} = 2^{2-1} = 2^1 = 2, \ a_{23} = 2^{3-1} = 2^2 = 4, \ a_{24} = 2^{4-1} = 2^3 = 8 \\ a_{31} = 3^{1-1} = 3^0 = 1, \ a_{32} = 3^{2-1} = 3^1 = 3, \ a_{33} = 3^{3-1} = 3^2 = 9, \ a_{34} = 3^{4-1} = 3^3 = 27 \\ a_{41} = 4^{1-1} = 4^0 = 1, \ a_{42} = 4^{2-1} = 4^1 = 4, \ a_{43} = 4^{3-1} = 4^2 = 16, \ a_{44} = 4^{4-1} = 4^3 = 64 \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

2. In each part below, find A, by use of the given information:

(7 pts.) **a)** $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ (Hint: $(A^T)^{-1} = (A^{-1})^T$)

$$(5A^T)^{-1} = \frac{1}{5}(A^T)^{-1} = \frac{1}{5}(A^{-1})^T = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} \Rightarrow (A^{-1})^T = 5\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -15 & -5 \\ 25 & 10 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -15 & 25 \\ -5 & 10 \end{bmatrix} \Rightarrow A = \frac{1}{-150+125}\begin{bmatrix} 10 & -25 \\ 5 & -15 \end{bmatrix} = \frac{1}{-25}\begin{bmatrix} 10 & -25 \\ 5 & -15 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} -2/5 & 1 \\ -1/5 & 3/5 \end{bmatrix}$$

(7 pts.) **b)** $(I+2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} \Rightarrow I + 2A = \frac{1}{-5-8}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = -\frac{1}{13}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix}$

$$\Rightarrow 2A = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2A = \begin{bmatrix} -18/13 & 2/13 \\ 4/13 & -12/13 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{bmatrix}$$

3. Given that $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

Find the Determinants: (4 pts. each)

a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -(-(-6)) = -6$

$R_1 \leftrightarrow R_2$

and

$R_2 \leftrightarrow R_3$

b) $\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix} = 2^3 \cdot (-6) = 8(-6)$

$= -48$

c) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4)(-6)$

$= 72$

d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = (-3)(-6) = 18$

$(-3)R_1 \to R_1$

$-4R_2 + R_3 \to R_3 \to$ doesn't change the determinant

e) $|A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{-6} = -\dfrac{1}{6}$

f) $|2A^{-1}| = 2^3 |A^{-1}| = 8\left(-\frac{1}{6}\right) = -\frac{4}{3}$

4. Consider the matrices;

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(4 pts.) **a)** Calculate $\det(A)$.

$$\det A = 1\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} + \cancel{0 \,|\;|}$$

w.r.t. R_3

$$= (2+3) - (1+3) = 5 - 4 = 1$$

Answer:

$\det(A) = 1$

(10 pts.) **b)** Find A^{-1} using the $Adj(A)$ matrix.

$$Adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} -1 & -(-1) & 0 \\ -(-3) & -3 & -(-1) \\ 5 & -(4) & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

$$A^{-1} = \frac{1}{\underbrace{\det A}_{1}} Adj(A)$$

Answer:

$$A^{-1} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

(5 pts.) **c)** Use the matrix A^{-1} found in part b) to solve the system $AX = B$.

$$X = A^{-1}B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$$

Answer:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$$

(20 pts.) **5.** If possible, solve the following linear system by using Cramer's rule: (only Cramer's rule method is accepted)

BE CAREFUL!

$$\left.\begin{cases} 3x_2 - 2x_1 - x_3 = 1 \\ x_1 - x_3 + 2x_2 = 4 \\ -2x_1 + 3 + x_3 = x_2 \end{cases}\right\} \Rightarrow \left.\begin{array}{r} -2x_1 + 3x_2 - x_3 = 1 \\ x_1 + 2x_2 - x_3 = 4 \\ -2x_1 - x_2 + x_3 = -3 \end{array}\right\} \Rightarrow AX = B$$

where $A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

$$\det(A) = -2\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -2(2-1) - 3(1-2) - (-1+4)$$
$$= -2(1) - 3(-1) - (3) = -2+3-3 = -2$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{-2} = \frac{1(2-1) - 3(4-3) - 1(-4+6)}{-2} = \frac{-4}{-2} = 2$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{-2} = \frac{-2(4-3) - 1(1-2) - 1(-3+8)}{-2} = \frac{-6}{-2} = 3$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{-2} = \frac{-2(-6+4) - 3(-3+8) + 1(-1+4)}{-2} = \frac{-8}{-2} = 4$$

$$\boxed{\underline{\text{Soln.:}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}$$

6. Using **Row Reduction**, **find all solutions** (if any) of the following systems. In each case, state whether there is a unique solution, no solution, or infinitely many solutions (depending on one or more parameters).

(7 pts.) **a)**
$$\begin{aligned} 2x + y - z &= 2 \\ 2x + z &= 3 \\ 3x - y &= 0 \end{aligned} \quad \begin{bmatrix} 2 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ 3 & -1 & 0 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_1 \to R_1]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & -1 & 2 & 1 \\ 3 & -1 & 0 & 0 \end{array}\right] \xrightarrow[-R_2 \to R_2]{-3R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & -2 & -1 \\ 0 & -\frac{5}{2} & \frac{3}{2} & -3 \end{array}\right] \xrightarrow[\frac{5}{2}R_2+R_3 \to R_3]{-\frac{1}{2}R_2+R_1 \to R_1}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -\frac{7}{2} & -\frac{11}{2} \end{array}\right] \xrightarrow{-\frac{2}{7}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & \frac{11}{7} \end{array}\right] \xrightarrow[2R_3+R_2 \to R_2]{-\frac{1}{2}R_3+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{7} \\ 0 & 1 & 0 & \frac{15}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{array}\right]$$

unique soln:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/7 \\ 15/7 \\ 11/7 \end{bmatrix}$$

(7 pts.) **b)**
$$\begin{aligned} x - z &= 2 \\ x + y - 3z &= 3 \\ -2y + 4z &= -2 \end{aligned} \quad \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 1 & 1 & -3 & 3 \\ 0 & -2 & 4 & -2 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \text{infinitely many solutions}$$

$$\Rightarrow \begin{cases} x - z = 2 \\ y - 2z = 1 \end{cases} \Rightarrow \begin{aligned} x &= 2 + z \\ y &= 1 + 2z \end{aligned}$$

Let $z = r \Rightarrow$ Soln: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ one parameter family of solns.

(7 pts.) **c)**
$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= -9 \\ 4x + y + 2z &= 3 \end{aligned} \quad \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & -9 \\ 4 & 1 & 2 & 3 \end{array}\right] \xrightarrow[-4R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -21 \\ 0 & -7 & 14 & -13 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -21 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

$0 \neq 8$

$\boxed{\text{No solution}}$