ÇANKAYA UNIVERSITY

Department of Mathematics

**MATH 106 - Business Mathematics II**

**2018-2019 Spring**

# FINAL EXAMINATION

20.05.2019, 10:00

-SOLUTIONS-

| Question | Grade | Out of |
|---|---|---|
| 1 | | 8 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 12 |
| 5 | | 36 |
| 6 | | 10 |
| 7 | | 4 |
| 8 | | 4 |
| 9 | | 12 |
| Total | | 106 |

**STUDENT NUMBER:**

**NAME-SURNAME:**

**SIGNATURE:**

**DURATION:** 100 minutes

**IMPORTANT NOTES:**

**1)** Please make sure that you have written your student number and name above.

**2)** Check that the exam paper contains 9 problems.

**3)** Show all your work. No points will be given to correct answers without reasonable work.

(8 pts.) **1.** Find the values of $x$ so that $\det(A) \neq 0$ for the matrix $A = \begin{bmatrix} x^3 & x^2 & x \\ 4 & -8 & 1 \\ 2 & 4 & -2 \end{bmatrix}$.

$\det(A) = x^3(16-4) - x^2(-8-2) + x(16+16) = 12x^3 + 10x^2 + 32x$

$\det(A) = 0 \Rightarrow 2x(\underbrace{6x^2+5x+16}) = 0$

$x = 0 \Leftarrow 0 \qquad b^2-4ac = 25-4(6)(16) < 0 \Rightarrow$ no real roots

So $\det(A)=0$ holds only when $x=0$.

$\Rightarrow$ for $\boxed{x \neq 0 \Rightarrow \det(A) \neq 0}$ (i.e. for $x \in \mathbb{R} \setminus \{0\}$ values $\det A \neq 0$)

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(10 pts.) **2.** If possible, solve the following linear system by using Cramer's rule: (only Cramer's rule method is accepted)

**BE CAREFUL!**

$$\left.\begin{cases} 3x + z + y = 3 \\ 2y + 2x + 1 = -5z \\ x - 2 - 3y = 4z \end{cases}\right\} \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & 1 & 3 \\ 2 & 2 & 5 & -1 \\ 1 & -3 & -4 & 2 \end{array}\right]$$

$A$ $B$

$|A| = 3(-8+15) - 1(-8-5) + 1(-6-2)$
$= 3(7) + 13 - 8 = \boxed{26}$

$$x = \frac{\begin{vmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \\ 2 & -3 & -4 \end{vmatrix}}{26} = \frac{3(-8+15) - 1(4-10) + 1(3-4)}{26} = \frac{21+6-1}{26} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 3 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & -4 \end{vmatrix}}{26} = \frac{3(4-10) - 3(-8-5) + 1(4+1)}{26} = \frac{-18+39+5}{26} = 1$$

$$z = \frac{\begin{vmatrix} 3 & 1 & 3 \\ 2 & 2 & -1 \\ 1 & -3 & 2 \end{vmatrix}}{26} = \frac{3(4-3) - 1(4+1) + 3(-6-2)}{26} = \frac{3-5-24}{26} = -1$$

$\boxed{\text{Solution: } (x,y,z) = (1,1,-1)}$

(10 pts.) **3.** If the <u>Augmented matrix</u> of a system of linear equations is given by

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 4 \end{array}\right]$$

using elementary row operations, **reduce the above matrix** and **find the solution (if any)** of the corresponding system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 4 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -7 \\ 0 & -2 & -2 & -2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3 \to R_3}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -7 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[-R_3+R_1 \to R_1]{2R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 2 & -5 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & \frac{-5}{2} \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -5/2 \\ 0 & 1 & 0 & 7/2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -5/2 \end{array}\right]$$

<u>Soln.:</u> $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7/2 \\ -5/2 \end{bmatrix}$

**4.** In each part below, find $A$, by use of the given information:

(6 pts.) **a)** $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ (Hint: $(A^T)^{-1} = (A^{-1})^T$)

$$(5A^T) = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{(-6+5)}\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$$

$$A^T = \frac{1}{5}\begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -2/5 & -1/5 \\ 1 & 3/5 \end{bmatrix}$$

$$A = \begin{bmatrix} -2/5 & -1/5 \\ 1 & 3/5 \end{bmatrix}^T = \begin{bmatrix} -2/5 & 1 \\ -1/5 & 3/5 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} -2/5 & 1 \\ -1/5 & 3/5 \end{bmatrix}$$

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(6 pts.) **b)** $(I+2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} \Rightarrow I+2A = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{-5-8}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix}$

$$\Rightarrow I+2A = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} -5/13-1 & 2/13 \\ 4/13 & 1/13-1 \end{bmatrix} = \begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix} \Rightarrow A = \frac{1}{2}\begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{bmatrix}$$

**5.** Evaluate the following integrals :

(9 pts.) a) $\int \frac{2}{t^2}\sqrt{\frac{1}{t}+9}\, dt = -2\int \sqrt{u}\, du = -2\,\frac{u^{3/2}}{3/2} + C$

$u = \frac{1}{t}+9$

$du = \frac{-1}{t^2}\,dt$

$= \boxed{-\frac{4}{3}\left(\frac{1}{t}+9\right)^{3/2} + C}$

(9 pts.) b) $\int \sqrt[3]{x}\, e^{\sqrt[3]{x^4}}\, dx = \int e^{x^{4/3}} \cdot x^{1/3}\, dx$

$u = x^{4/3}$

$du = \frac{4}{3} x^{\frac{1}{3}}\, dx$

$\frac{3}{4} du = x^{1/3}\, dx$

$= \int e^u \cdot \frac{3}{4}\, du = \frac{3}{4} e^u + C$

$= \boxed{\frac{3}{4} e^{\sqrt[3]{x^4}} + C}$

(9 pts.) c) $\int y^3 \ln y\, dy = \frac{y^4}{4}\cdot \ln y - \frac{1}{4}\int y^3\, dy = \frac{y^4}{4}\cdot \ln y - \frac{1}{4}\,\frac{y^4}{4} + C$

$u = \ln y \Rightarrow du = \frac{1}{y}\, dy$

$dv = y^3\, dy \Rightarrow v = \frac{y^4}{4}$

$= \boxed{\frac{y^4}{4}\cdot \ln y - \frac{y^4}{16} + C}$

$\Big($ <u>or</u> $= \boxed{\frac{y^4}{4}\left[\ln y - \frac{1}{4}\right] + C}$

(9 pts.) d) $\int \left(e^x + \frac{x}{2}\right)^2 dx = \int \left(e^{2x} + x\cdot e^x + \frac{x^2}{4}\right) dx$

$= \int e^{2x}\, dx + \underbrace{\int x\cdot e^x\, dx}_{} + \int \frac{x^2}{4}\, dx = \frac{e^{2x}}{2} + \left[x\cdot e^x - \int e^x\, dx\right] + \frac{x^3}{12} + C$

$\left(\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x\, dx \Rightarrow v = e^x \end{array}\right)$

$= \boxed{\frac{e^{2x}}{2} + x e^x - e^x + \frac{x^3}{12} + C}$

(10 pts.) **6.** Find the area of the region bounded by the graph of $y = \dfrac{6(x^2+1)}{(x+2)^2}$ and the $x$-axis, from $x = 0$ to $x = 1$. (No need to sketch the region!)

$$\int_0^1 \frac{6(x^2+1)}{(x+2)^2}\,dx = \int_0^1 \left[6 - 6\,\frac{4x+3}{(x+2)^2}\right]dx = 6x - 6\int_0^1 \left[\frac{4}{x+2} - \frac{5}{(x+2)^2}\right]dx$$

$$= \left[6x - 24\ln|x+2| - 30\left(\frac{1}{x+2}\right)\right]\Big|_0^1$$

$$= 6 - 24(\ln 3 - \ln 2) - 30\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= 6 - 24\ln\left(\frac{3}{2}\right) - 30\left(\frac{-1}{6}\right)$$

$$= \boxed{11 - 24\ln\left(\frac{3}{2}\right)}$$

$$(\text{or}) = \boxed{11 + 24\ln\left(\frac{2}{3}\right)}$$

$$\begin{array}{r|l} 6x^2+6 & x^2+4x+4 \\ \hline -6x^2 \mp 24x \mp 24 & 6 \\ \hline -24x-18 & \end{array}$$

$$\frac{4x+3}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

$$\left.\begin{array}{l} A=4 \\ B=-5 \end{array}\right\}$$

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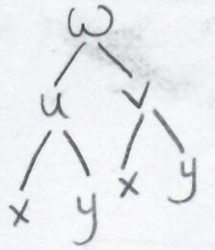
(4 pts.) **7.** Let $w = w(u,v)$, $u = xy$, $v = x - y$. Suppose for $x = 1$, $y = 2$, $w_u(1,2) = 1$, $w_v(1,2) = 3$ and $w(2,-1) = 5$. Evaluate $w_x(1,2)$ and $w_y(1,2)$.




$$\omega_x = \frac{\partial\omega}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\omega}{\partial v}\cdot\frac{\partial v}{\partial x} = (\omega_u)(y) + (\omega_v)\cdot(1)$$

$$\omega_x(1,2) = (\omega_u(1,2))(2) + (\omega_v(1,2))\cdot(1) = (1)(2) + (3)(1) = 5$$

$$\omega_y = \frac{\partial\omega}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\omega}{\partial v}\cdot\frac{\partial v}{\partial y} = (\omega_u)\cdot(x) + (\omega_v)\cdot(-1)$$

$$\omega_y(1,2) = (\omega_u(1,2))(1) + (\omega_v(1,2))\cdot(-1)$$

$$= (1)(1) + (3)(-1) = 1 - 3 = -2$$

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(4 pts.) **8.** Find $f_{zy}$ for $f(x,y,z) = yz\ln(xy)$.

$$f_z = \frac{\partial f}{\partial z} = y\cdot\ln(xy)$$

$$f_{zy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y\,\partial z} = (1)(\ln(xy)) + \overbrace{(y)\left(\frac{x}{xy}\right)}^{1} = \boxed{\ln(xy) + 1}$$

(12 pts.) **9.** Find **four critical points** of the function

$$f(x,y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1$$

and **classify them as relative maxima, relative minima or saddle points** using the second derivative test.

$f_x = \frac{1}{3}(3x^2) - 2(2x) = x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x=0,\ x=4$

$f_y = \frac{1}{3}(24y^2) - 2(2y) = 8y^2 - 4y = 0 \Rightarrow 4y(2y-1) = 0 \Rightarrow y=0,\ y=\frac{1}{2}$

Crit. pts.: $\boxed{(0,0), (0,\tfrac{1}{2}), (4,0), (4,\tfrac{1}{2})}$

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$f_{xx} = 2x-4,\quad f_{xy} = 0 = f_{yx},\quad f_{yy} = 16y-4$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (2x-4)(16y-4) = 8(x-2)(4y-1)$

$D(0,0) = 8(-2)(-1) = 16 > 0,\ f_{xx}(0,0) = -4 < 0 \Rightarrow \boxed{(0,0) \text{ rel.max.pt.}}$

$D(0,\frac{1}{2}) = 8(-2)(2-1) = -16 < 0 \Rightarrow \boxed{(0,\tfrac{1}{2}) \text{ saddle pt.}}$

$D(4,0) = 8(4-2)(0-1) = 8(2)(-1) = -16 < 0 \Rightarrow \boxed{(4,0) \text{ saddle pt.}}$

$\left.\begin{array}{l} D(4,\frac{1}{2}) = 8(4-2)(2-1) = 8 \cdot 2 \cdot 1 = 16 > 0 \\ f_{xx}(4,\frac{1}{2}) = 2(4) - 4 = 8 - 4 = 4 > 0 \end{array}\right\} \Rightarrow \boxed{(4,\tfrac{1}{2}) \text{ rel. min. pt.}}$