Çankaya University
Department of Mathematics
2017 - 2018 Spring Semester

MATH 108 - Mathematics for Business and Economics - II

Second Midterm Examination

1) This question has two unrelated parts.

a) Find the first partial derivatives, f_x, f_y of the following function:

$$f(x,y) = x^2 \ln y + y^3 e^x$$

b) Find f_{xx} and f_{yy} of the the following function:

$$f(x,y) = x^3 y^4 + \ln(xy) + \frac{x}{y}$$

2) This question has two unrelated parts.

a) Find z_s and z_t of the following function using chain rule where $x = st$ and $y = 2s - 3t$:

$$z = x^3 y + x^2 y^2$$

b) Let the equation is given by $z^3 + xy^2 + y^3 z^4 = 6$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

3) Find and classify the critical points of $f(x,y) = 16x^3 + 16x^2 - 4xy + y^2$ for relative extrema by using second derivative test.

4) Use Lagrange multipliers method to find critical points (maximum, minimum) of $f(x,y) = 4x + y$ subject to the constraint $x^2 + y = 24$.

5) Evaluate the following double integrals.

a) $\displaystyle\int_0^3 \int_0^1 y e^x \, dx dy$

b) $\displaystyle\int_1^5 \int_0^1 \frac{2x}{y(x^2+1)} \, dx dy$

Answers

1)

a)
$$f_x = 2x\ln y + y^3 e^x$$
$$f_y = \frac{x^2}{y} + 3y^2 e^x$$

b)
$$f_x = 3x^2y^4 + \frac{1}{x} + \frac{1}{y}$$
$$f_{xx} = 6xy^4 - \frac{1}{x^2}$$

$$f_y = 4x^3y^3 + \frac{1}{y} - \frac{x}{y^2}$$
$$f_{yy} = 12x^3y^2 - \frac{1}{y^2} + \frac{2x}{y^3}$$

2)

a)
$$\begin{aligned} z_s &= z_x \cdot x_s + z_y \cdot y_s \\ &= \big(3x^2y + 2xy^2\big)t + \big(x^3 + 2x^2y\big)2 \end{aligned}$$

$$\begin{aligned} z_t &= z_x \cdot x_t + z_y \cdot y_t \\ &= \big(3x^2y + 2xy^2\big)s + \big(x^3 + 2x^2y\big)(-3) \end{aligned}$$

b)
$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} \\ &= -\frac{y^2}{3z^2 + 4y^3z^3} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} \\ &= -\frac{2xy + 3y^2z^4}{3z^2 + 4y^3z^3} \end{aligned}$$

3) $$\begin{aligned} f_x &= 48x^2+32x-4y=0 \\ f_y &= -4x+2y=0 \quad \Rightarrow \quad y=2x \end{aligned}$$

Using $y=2x$ in the first equation, we obtain:

$$\begin{aligned} 48x^2+32x-8x &= 0 \\ 48x^2+24x &= 0 \\ 24x(2x+1) &= 0 \end{aligned}$$

$$x=0 \quad \Rightarrow \quad y=0 \quad \text{OR} \quad x=-\frac{1}{2} \quad \Rightarrow \quad y=-1$$

There are two critical points: $(0,0)$ and $(-1/2,-1)$.

$$\begin{aligned} f_{xx} &= 96x+32 \\ f_{yy} &= 2 \\ f_{xy} &= -4 \end{aligned}$$

$$\begin{aligned} \Delta = D(x,y) &= f_{xx}f_{yy}-\big(f_{xy}\big)^2 \\ &= \big(96x+32\big)\cdot 2-\big(-4\big)^2 \\ &= 192x+48 \end{aligned}$$

For $(0,0)$, $\quad \Delta = 48>0$ and $A=f_{xx}(0,0)=32>0$ so $(0,0)$ is a local minimum.

For $(-1/2,-1)$, $\quad \Delta=-48<0$ so $(-1/2,-1)$ is a saddle point.

4) $F(x,y,\lambda)=4x+y-\lambda(x^2+y-24)$

$F_x=F_y=F_\lambda=0$

OR $\quad f_x=g_x, \quad f_y=g_y, \quad g=$constant.

$$\Rightarrow \quad \begin{aligned} 4-2x\lambda &= 0 \\ 1-\lambda &= 0 \\ x^2+y-24 &= 0 \end{aligned}$$

$\lambda=1$

$\Rightarrow \quad x=2$

$\Rightarrow \quad 4+y-24=0$

$\Rightarrow \quad y=20$

The only critical point is $(2,20)$.

5)

a) $$\int_0^3\int_0^1 ye^x\,dxdy = \int_0^3 y\left(e^x\Big|_0^1\right)dy$$
$$= \int_0^3 y\left(e^1-e^0\right)dy$$
$$= \left(e-1\right)\int_0^3 y\,dy$$
$$= \left(e-1\right)\frac{y^2}{2}\Big|_0^3$$
$$= \frac{9}{2}(e-1)$$

b) Using $u = x^2+1$ and $du = 2x\,dx$, we obtain:

$$\int\frac{2x\,dx}{x^2+1} = \int\frac{du}{u} = \ln|u| = \ln\left(x^2+1\right)$$

$$\int_1^5\int_0^1\frac{2x}{y(x^2+1)}\,dxdy = \int_1^5\frac{1}{y}\left(\ln\left(x^2+1\right)\Big|_0^1\right)\,dy$$
$$= \int_1^5\frac{1}{y}\left(\ln 2-\ln 1\right)\,dy$$
$$= \ln 2\int_1^5\frac{dy}{y}$$
$$= \ln 2\cdot\ln|y|\Big|_1^5$$
$$= \ln 2\cdot(\ln 5-\ln 1)$$
$$= \ln 2\cdot\ln 5$$